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Forced Oscillations of Singular Dynamical Systems with an Application to the Restricted Three Body Problem

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We consider forced dynamical systems with two degrees of freedom having singular potentials and we prove existence of infinitely many classical (noncollision) periodic solutions. These solutions have a prescribed rotation behavior with respect to the singularities and a prescribed period (the same of the systems). They are obtained variationally as minima of a suitable functional on open subsets of a Hilbert space. This investigation was motivated by the elliptic restricted three body problem with arbitrary masses of the two primaries. For that problem we obtain infinitely many distinct "generalized" periodic solutions (i.e., solutions which possibly experience collisions). © 1991 Academic Press, Inc.

1. INTRODUCTION AND MAIN RESULTS

(a) *Motivation and Introduction*

Our aim is to establish the existence of infinitely many forced oscillations for second order systems of two degrees of freedom

$$\ddot{x} = \frac{\partial}{\partial x} U(t, x) \tag{1.1}$$

with singular potentials. (Here $\ddot{x}$ denotes d^2x/dt^2 and $\partial U/\partial x = (\partial U/\partial x_1, \partial U/\partial x_2)$).

The model problem which raised our interest is the (circular) restricted three body problem. It consists of the study of the motion of a body of negligible mass (the satellite) in a gravitational field due to two bodies (the primaries) of positive masses μ and ν, which are assumed to revolve in circular orbits around their common center of mass. The satellite does

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not influence the motion of the primaries and is assumed to move (initially and therefore forever) in the plane determined by their orbits.

The equations of motion for this system are obtained as follows: one initially writes down the equations for a gravitational three body system, where the bodies are thought of as point masses. One then considers the limiting case in which one of the masses is equal to zero, the effect of which is that the equations of motion for the two bodies of positive mass are uncoupled from the third one. As equations governing a two body system, they can easily be integrated; at this step one selects between the possible motions of the primaries the "circular" ones, as specified above. In an inertial coordinate system the motion of the satellite is governed by Eq. (1.1), where

$$U(t, x) = \frac{\mu}{|x - P_\mu(t)|} + \frac{\nu}{|x - P_\nu(t)|}. \tag{1.2}$$

Here the constant of gravitation, as well as the sum of the masses μ and ν, is normalized to 1 and, if ω denotes the frequency of rotation of the primaries,

$$P_\mu(t) = (-\nu \cos \omega t, \ -\nu \sin \omega t)$$
$$P_\nu(t) = (\mu \cos \omega t, \mu \sin \omega t).$$

What immediately springs to the eyes is that this potential admits singularities: indeed $U(t, x)$ is not defined for $x = P_\mu(t)$ and $x = P_\nu(t)$ and it tends to infinity when $x - P_\mu(t)$ or $x - P_\nu(t)$ tends to zero.

Most commonly this problem is studied in a rotating coordinate system, where the primaries are fixed and an integral of motion (the Jacobi integral) exists. Our choice to work in a nonrotating system allows us to include in our investigations also more general problems, for example, the elliptic planar restricted three body problem. This problem is more general than the circular one in that the primaries are assumed to move on elliptic, rather than circular, orbits around their center of mass. We point out that for the elliptic restricted three body problem it is not possible to pass to an uniformly rotating system where an integral of motion exists.

With these examples in mind, we consider time dependent systems of two degrees of freedom:

$$\ddot{x} = \frac{\partial}{\partial x} U(t, x) \qquad (t, x) \in \mathbb{R}^3 \setminus S, \tag{1.3}$$

defined on the open set $\mathbb{R}^3 \setminus S$.

The set S represents the singularity set for U and is defined as follows.

We suppose that N ($N \in \mathbb{N}^*$) periodic functions of period $T > 0$ are given

$$P_j \in C^1(\mathbb{R}, \mathbb{R}^2), \qquad P_j(t+T) = P_j(t) \qquad t \in \mathbb{R},\ 1 \leqslant j \leqslant N, \tag{1.4}$$

satisfying

$$P_k(t) \neq P_j(t) \qquad \text{all } t \in \mathbb{R},\ 1 \leqslant k \neq j \leqslant N. \tag{1.5}$$

The set $S \subset \mathbb{R}^3$ is then determined as

$$S := \{(t, P_1(t)) \cup \cdots \cup (t, P_N(t)) : t \in \mathbb{R}\} \subset \mathbb{R}^3. \tag{1.6}$$

For the potential the following assumptions are suggested by the problem above:

$$\begin{aligned}
&\text{(i)} \quad U \in C^2(\mathbb{R}^3 \backslash S, \mathbb{R}),\\
&\text{(ii)} \quad U(t+T, x) = U(t, x) \qquad \text{for all} \quad (t, x) \in \mathbb{R}^3 \backslash S,\\
&\text{(iii)} \quad U(t, x) > 0 \qquad \text{for all} \quad (t, x) \in \mathbb{R}^3 \backslash S,\\
&\text{(iv)} \quad U(t, x), \frac{\partial}{\partial x} U(t, x) \to 0 \text{ uniformly in } t \text{ as } |x| \to \infty,\\
&\text{(v)} \quad U(t, x) \to +\infty \qquad \text{as} \quad (t, x) \to S.
\end{aligned} \tag{1.7}$$

In addition to (1.7) the following assumption will be needed in some of our results: there exists a function $W \in C^1(\mathbb{R}^3 \backslash S, \mathbb{R})$, satisfying

$$\begin{aligned}
&\text{(i)} \quad W(t+T, x) = W(t, x) \qquad \text{for all} \quad (t, x) \in \mathbb{R}^3 \backslash S,\\
&\text{(ii)} \quad W(t, x) \to -\infty \qquad \text{as} \quad (t, x) \to S,\\
&\text{(iii)} \quad |\nabla W(t, x)|^2 \leqslant a_1 U(t, x) + a_2 \qquad a_1, a_2 \text{ positive constants.}
\end{aligned} \tag{1.8}$$

(∇W denotes $(\partial W/\partial t, \partial W/\partial x_1, \partial W/\partial x_2)$.)

The aim is to find T-periodic solutions of (1.3). They will have a prescribed rotation behavior with respect to the singularities $P_j(t)$, $1 \leqslant j \leqslant N$. For this purpose, we associate to every continuous closed curve $x \in C(\mathbb{R}, \mathbb{R}^2)$, satisfying

$$x(t+T) = x(t) \qquad \text{all} \quad t, \tag{1.9}$$

$$x(t) \neq P_j(t) \qquad \text{all} \quad t,\ 1 \leqslant j \leqslant N, \tag{1.10}$$

N maps (here and henceforth we will identify S^1 with $\mathbb{R}/[0, T]$):

$$\xi_j(x) \colon S^1 \to S^1 \qquad 1 \leqslant j \leqslant N. \tag{1.11}$$

They are defined as follows:

$$\xi_j(x)(t) := \frac{x(t) - P_j(t)}{|x(t) - P_j(t)|}. \tag{1.12}$$

The $\xi_j(x)$ are well defined (the denominators never vanish) and continuous functions. Denoting the mapping degree of these circle maps by $\deg(\xi_j(x))$, we associate to the curve x the vector:

$$\deg(x) := (\deg(\xi_1(x)), ..., \deg(\xi_N(x))) \in \mathbb{Z}^N. \tag{1.13}$$

(b) *Main Results*

Our first result is then the following:

THEOREM 1. *Assume that U satisfies* (1.7) and (1.8). *Then the system*

$$\ddot{x} = \frac{\partial}{\partial x} U(t, x) \qquad (t, x) \in \mathbb{R}^3 \backslash S, \tag{1.3}$$

admits for every prescribed $k \in \mathbb{Z}^N$, $k \neq 0$, a T-periodic solution x with $\deg(x) = k$. *In other words for every prescribed $k = (k_1, k_2, ..., k_N) \in \mathbb{Z}^N$,* $k \neq (0, 0, ..., 0)$, *there is a solution of* (1.3), *which is periodic of period T and which, in an interval of time of length T, winds $|k_j|$ times around $P_j(t)$ $(0 \leqslant t \leqslant T)$ for each $1 \leqslant j \leqslant N$, counterclockwise or clockwise, depending on the sign $+$ or $-$ of k_j.*

Remark. We can actually say more: indeed, as will be evident by the proof, there is a solution in each homotopy class (in $[0, T] \times \mathbb{R}^2 \backslash \{S \cap ([0, T] \times \mathbb{R}^2)\}$) of curves $(t, x(t))$ with values in $\mathbb{R}^3 \backslash S$ and x T-periodic.

In the special case of two singularities, one deduces:

COROLLARY 2. *Consider a restricted three body problem with potential satisfying* (1.7) *and* (1.8). *Then, for every prescribed $k = (k_1, k_2) \in \mathbb{Z}^2$,* $k \neq (0, 0)$, *there is a solution which is periodic of period T and according to which, in an interval of time of length T, the satellite winds $|k_1|$ times around one and $|k_2|$ times around the other of the primaries, counterclockwise or clockwise, depending on the sign $+$ or $-$ of k_1 and k_2.*

In view of our interest for the restricted three body problem arising in celestial mechanics a broader notion of solution ("generalized solution") is introduced in Section 4. The origin of this notion is illustrated in the last part of this paragraph. Roughly speaking a generalized T-periodic solution of (1.3) coincides with a classical solution $x(t)$ for each t, except on a subset of measure zero, on which collisions with one of the P_j's take place. In other words a generalized solution (which in particular may be a classical solution) is obtained by gluing together different solutions x_i, $i \in J \subset \mathbb{N}$: if the cardinality of J is strictly greater than 1, a collision of the solution x_i is followed by an ejection of the solution x_j for $i, j \in J$. We prove:

THEOREM 3. *Let $T>0$ be the first time after which in the elliptic restricted three body problem of celestial mechanics (described above) the two primaries occupy the same position. Then there exist infinitely many distinct generalized T-periodic solutions.*

Our approach to the problem is variational. A main advantage provided by such an approach is that no smallness assumptions are required for the masses μ and ν of the primaries. This is, as far as we know, in contrast to all the analytical results available on periodic orbits for the (circular) restricted three body problem (rtbp), apart from one due to Conley [1] and described below.

(c) *A Short Historical Review*

We recall in this connection a few (among the principal) lines along which existence of periodic motions in the rtbp has been studied. The amount of research stimulated by this problem is so large that we cannot even think of giving a complete survey here. A source of references devoted to the rtbp is provided by Szebehely [2]. For a treatise on Celestial Mechanics we refer to Siegel and Moser [3] (see also Moser [4]).

The interest for periodic orbits in the rtbp was inaugurated and emphasized by Poincaré [5] (even if some particular closed orbits, namely the collinear and the equilateral ones which are equilibria in a rotating coordinate system, were discovered already by Euler [6] and Lagrange [7]). In particular Poincaré developed the method of analytical continuation for the case where one of the primaries has a very small mass compared to the other, and the satellite moves close to the larger mass point. This case can be looked at as a perturbation of the Kepler problem. In [5] Poincaré established the existence of solutions "continuing" from the circular ones and from the elliptical ones of the Kepler problem (*solutions de la première sorte* and *de la deuxième sorte*, respectively), lying in the plane of the primaries and of other continuing solutions, whose inclination on the primaries plane is different from zero (*solutions de la troisième sorte*). He also developed a different approach (again requiring a restriction on the mass ratio) and formulated a fixed point theorem for annulus mappings, which determines at once infinitely many orbits and which was proved by Birkhoff [8].

In [1] Conley studied the problem with arbitrary positive masses of the primaries for large negative values of the Jacobian integral, in such a way that two compact Hill's regions surrounding the primaries and separating them exist. He assumed the satellite was moving close to one of the primaries (in particular that of smaller mass) and he proved, by means of an elaborated application of the Poincaré–Birkhoff Theorem, existence of infinitely many long-period closed orbits.

Other results by Conley [9–11] refer to the case when the Jacobian constant is fixed and slightly greater than that of the collinear equilibrium point (L_2) between the two primaries. In this case the satellite can travel from one primary to the other and an unstable closed orbit is known to exist near L_2. The results in [9–11] are obtained by means of a local analysis in a neighborhood of the above mentioned unstable orbit. They rely on the construction of a relative surface of section (which requires a smallness assumption for one of the primaries masses) and of an annulus mapping which makes an infinite twist.

In [9] infinitely many closed orbits (winding around the largest primary and passing near the unstable orbit) are found. The existence of such orbits can occur in two cases. One case involves the use of nondegenerate homoclinic orbits while the other case requires an application of the Poincaré–Birkhoff theorem. (Results in this spirit for other values of the Jacobi constant are derived in McGehee's Thesis [12]). In contrast to [9], in [10] orbits (not necessarily closed) are found which travel from the far side of one primary to the far side of the other infinitely often. Taking advantage of this result in [11] a scheme is suggested for *designing* periodic orbits transiting between the earth and the moon (the two primaries) and winding around each of them a prescribed number of times.

An approach of variational nature (based on Maupertuis Least Action Principle and leading to a geodesical interpretation) is at the origin of a criterion to detect the presence of periodic orbits, which is due to Whittaker. Specifically this criterion was first given for systems of two degrees of freedom of the form $\ddot{x} = (\partial/\partial x)\, U(x)$ [13] and then in [14] it was extended to the rtbp, whose equations in a rotating coordinate system are

$$\ddot{x}_1 - 2\omega\dot{x}_2 = \frac{\partial}{\partial x_1}\, \Omega(x)$$

$$\ddot{x}_2 + 2\omega\dot{x}_1 = \frac{\partial}{\partial x_2}\, \Omega(x),$$

where

$$\Omega = \frac{1}{2}\omega^2(x_1^2 + x_2^2) + \frac{\mu}{((x_1 - \nu)^2 + x_2^2)^{1/2}} + \frac{\nu}{((x_1 + \mu)^2 + x_2^2)^{1/2}}.$$

Both papers [13] and [14] rely mostly on an intuitive idea and are missing mathematical rigor. The content of the first paper has been since placed upon a rigorous basis (and extended) by Signorini [15, 16] and Tonelli [17, 18] independently. A proof of the Whittaker criterion for more general systems (like the rtbp) is contained in Birkhoff [19].

The criterion is roughly as follows. With reference to a closed curve in the plane a certain expression K involving the curvature, the normal (to the curve) component of $\nabla\Omega$, and the value of the integral of motion is given:

$$K = \frac{2(h+\Omega)}{\rho} + 2\omega((2(h+\Omega))^{1/2}) + \nabla\Omega \cdot n.$$

Here h is the prescribed value of the Jacobi constant: $\frac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2) - \Omega = h$, ρ is the curvature radius, and n is the exterior normal of the curve.

If two curves γ_1 and γ_2 can be found which form the boundary of an annulus such that K is always negative along γ_1 and always positive along γ_2, then the existence of a simple closed orbit in the interior of the annulus can be deduced.

Strangely enough, it seems that nowhere an explicit application to the rtbp is carried out. We therefore point out here that the Whittaker criterion can be successfully used when the Jacobian constant is large negative and one of the masses of the primaries, μ, is very small (actually almost zero). Indeed one can take as curves bounding an annulus

γ_1 = the zero velocity curve bounding the component of the Hill's region surrounding the greatest primary ν,

γ_2 = a circle of small radius centered in ν.

As the reader can easily check, when the above assumptions are satisfied, $K|_{\gamma_1} < 0$ and $K|_{\gamma_2} > 0$.

(d) *Comments on the Variational Technique*

As announced above our approach to the problem is variational.

During the past few years a number of authors have investigated variationally the existence of periodic solutions (of prescribed period) of conservative dynamical systems

$$\ddot{q} = \nabla U(q) \tag{1.14}$$

having singular potentials (i.e., such that $U\colon \mathbb{R}^n \backslash S \to \mathbb{R}$, $U(q) \to +\infty$ if $q \to S$).[1] Under various assumptions on U, T-periodic solutions have been found as critical points of the Lagrangian functionals

$$\int_0^T \left\{ \frac{1}{2} |\dot{q}(t)|^2 + U(q(t)) \right\} dt$$

defined on suitable subsets of suitable loop spaces (see, e.g., [20–31], and their bibliographies).

One of the main difficulties in such a variational approach is due to the fact that ejection–collision orbits (namely orbits originating and dying at a singular point) are read just as periodic ones. However, solutions which experience collisions do not have the repetitive character of the classical periodic solutions; therefore, one would like to distinguish those solutions by the classical ones. In other words, having found the existence of some solutions, one needs an argument in order to recognize whether or not they are bounded away from the singularity. The problem can be successfully faced for certain singular potentials (strong force potentials; see Remarks I–III in Section 2) based on the work by Gordon [20, 21]. But it turns out to be a particularly hard task when gravitational potentials (e.g., $U(q) = 1/|q|$) are involved. For example, unlike what happens for other singular potentials, the Lagrangian functionals corresponding to gravitational potentials may attain finite values on collision orbits as well as on regular ones. This fact (a consequence of the very "steep" local growth rate near the singularity) can be easily verified in the Kepler problem:

$$\ddot{x} = \frac{\partial}{\partial x} \frac{1}{|x|} \qquad x \in \mathbb{R}^2 \backslash \{0\}. \tag{1.15}$$

We recall that for negative values of the energy the solutions of (1.15) (ellipses in the configuration space) occur in families, where all orbits have the same minimal period and the same energy. To these families belong also ejection–collision homothetic orbits. Moreover, the value of the Lagrangian functional is the same on all orbits (classical and collision ones) belonging to one of the above mentioned families. This already gives an insight to the difficulty of the problem of distinguishing collision orbits from regular ones in the case of gravitational potentials. Also according to further investigations this problem appears to a very deep one, at least when there are not symmetries allowing the use of suitable devices (this is the case for the Kepler problem, because the potential is radial). For that

[1] After this paper was written, we knew that more recently also N body-type problems have been considered [23, 24].

reason the notion of generalized solution (in relation to the study for gravitational potentials) has been introduced in [22] and then adopted in [23, 24] and in the present paper.

We point out finally the main differences between the quoted variational papers [20–31] and the present one. They concern the general setting, namely the properties of the potentials and singularities, as well as the techniques involved and the results.

In [25] (see also [26]) the singularity is typically given by the origin of $\mathbb{R}^n$, and the fact that central force fields are studied plays a crucial role. In this particular setting the authors can evaluate the infimum of the functional on the set of collision orbits; as a consequence they can get rid of the mentioned difficulty. We could not use the ideas of that paper to handle our problem. In [27] strong force potentials are considered. The singularity is given by a compact subset of $\mathbb{R}^n$ and the (existence and multiplicity) proofs rely on Morse type arguments. In [28] noncollision orbits are obtained for potentials $V(x)$, which behave like $-|x|^{-\alpha}$ $(\alpha>0)$ for x close to 0. Particularly crucial to the proof is an assumption on V which is not satisfied in the physical model we consider (the maximum of V has to be attained on the boundary of a starshaped, bounded open set surrounding the origin). In [29–31] existence is proved of one periodic solution (and multiplicity if V is autonomous or even) for strong force potentials. For the problems considered (without any symmetry assumption) we can find infinitely many periodic solutions.

The variational framework we adopt is related to that of [22]. In [22] dynamical systems with $n>2$ degrees of freedom and with compact (nonmoving) singularities are investigated and existence results about periodic orbits are given (multiplicity is guaranteed for autonomous systems, while one orbit is guaranteed for time dependent systems). For gravitational potentials in particular the authors find generalized solutions.

Peculiar to the present paper is the fact that the singularities (a finite number) themselves are periodically moving. As a consequence the systems we consider are forced. Nevertheless we guarantee existence of infinitely many distinct periodic solutions (in Section 3 also subharmonic solutions with minimal period kT are obtained), furnishing in addition a rotational characterization.

The question of whether the generalized orbits we find for the restricted three body problem are classical or not has remained open up to now (the same holds true, as far as we know, for the generalized orbits found in [22–24]).

2. Forced Oscillations for Systems with "Strong Force" Potentials

We first make some remarks concerning assumption (1.8).

Remarks.

(I) Assumption (1.8) concerns the local behavior of the potential U near the singularities. It is a generalization for problems with moving singularities of a condition (the "Strong Force" condition), which was introduced by Gordon in a pioneering work in [20]. In that paper Gordon studied autonomous systems

$$\ddot{x} = \frac{\partial}{\partial x} U(x),$$

whose potential U has singularities at a closed nonempty set (typically the origin of $\mathbb{R}^n$) and introduced the SF (Strong Force) condition in order to avoid difficulties inherent to collisions (see Section 1 above). We refer to [20, 21] for precise details.

We point out here that (1.8) is not satisfied by gravitational potentials (see, e.g., Remark (II) below). Namely, (1.8) is not verified by the systems we are mainly interested in. We study in this section systems for which (1.8) holds true and we obtain an auxiliary result. In the next section we will face systems of the form (1.3), for which (1.8) is not verified, by means of an approximating argument.

(II) Let $P(t)$ be a known function: $P \in C^1(\mathbb{R}, \mathbb{R}^2)$, $P(t) = (p_1(t), p_2(t)) \in \mathbb{R}^2$, such that $P(t) = P(t+T)$. Call $r = r(t, x) = |x - P(t)|$ and consider a gravitational-type potential

$$U(t, x) = \frac{1}{|x - P(t)|} = \frac{1}{r(t, x)}.$$

We show next that no function W, satisfying (1.8), can exist. Indeed, assume by contradiction that there exists a W satisfying (1.8). Introduce the function

$$V(t, x) = \hat{V}(r(t, x)) = \int_0^{r(t,x)} \left(a_1 \frac{1}{s} + a_2\right)^{1/2} ds.$$

Fix a t in $[0, T]$ and consider the corresponding singular point $P(t) \in \mathbb{R}^2$. Let $Q \in \mathbb{R}^2$ be a point different from $P(t)$: $Q \neq P(t)$.

Consider the line-segment in $\mathbb{R}^3$ connecting (t, Q) and $(t, P(t))$:

$$R := \{(t, x) = (t, \lambda P(t) + (1 - \lambda) Q) : 0 \leqslant \lambda \leqslant 1\}.$$

If (t, x) lies on R: $r(t, x) = (1-\lambda)\,|Q - P(t)|$. From this one deduces:

$$\begin{aligned}&\frac{d}{d\lambda} V(t, \lambda P(t) + (1-\lambda) Q))\\ &\qquad = (-1)\,|Q - P(t)|\,(a_1 U(t, \lambda P(t) + (1-\lambda) Q) + a_2)^{1/2}\end{aligned} \tag{2.2}$$

and

$$V(t, P(t)) = 0. \tag{2.3}$$

Since W satisfies (2.2), for every $0 \leqslant \lambda \leqslant 1$ the following estimate holds true:

$$\begin{aligned}&|W(t, \lambda P(t) + (1-\lambda) Q)) - W(t, Q)|\\ &\quad = \left| \int_0^\lambda \frac{d}{ds} W(t, sP(t) + (1-s) Q))\, ds \right|\\ &\quad = \left| \int_0^\lambda \left\langle \frac{\partial}{\partial x} W(t, sP(t) + (1-s) Q), P(t) - Q \right\rangle ds \right|\\ &\quad \leqslant |P(t) - Q| \int_0^\lambda \left| \frac{\partial}{\partial x} W(t, sP(t) + (1-s) Q) \right| ds\\ &\quad \leqslant |P(t) - Q| \int_0^\lambda (a_1 U(t, \lambda P(t) + (1-\lambda) Q) + a_2)^{1/2}\, ds\\ &\quad = |V(t, \lambda P(t) + (1-\lambda) Q)) - V(t, Q)|.\end{aligned}$$

When $\lambda \to 1$ the lhs of this inequality tends to infinity (in view of (1.8)), while the rhs tends to $|V(t, Q)| < +\infty$ (in view of (2.3)). But this is a contradiction!

In summary, a function W verifying (1.8) cannot exist if $U = 1/r(t, x)$. Hence a gravitational potential does not satisfy (1.8).

(III) Let us now consider

$$U(t, x) = \frac{1}{|x - P(t)|^2} = \frac{1}{(r(t, x))^2}$$

with $P(t)$ as in Remark (II), and let W be:

$$W(t, x) = \log r(t, x).$$

This function W is C^1 on $\mathbb{R}^3 \backslash S$ (namely for $x \neq P(t)$) and a computation shows that

$$|\nabla W(t, x)|^2 \leqslant \frac{1 + 2K^2}{(r(t, x))^2} = (1 + 2K^2)\, U(t, x),$$

where $K := \max_{1 \leqslant j \leqslant 2}\{\|p_j'\|_{L^\infty}\}$. Hence in this case assumption (1.8) is satisfied.

(IV) We show with an example how a function W can be constructed in a case where the singularity is given by two moving points and, of course, U behaves in a suitable way near the singularity. Let the singularity be given by

$$P(t) = ((p_1(t)), (p_2(t))) \quad \text{and} \quad Q(t) = ((q_1(t)), (q_2(t))) \in \mathbb{R}^2.$$

According to our general setting, we assume:

$$P(t+T) = P(t) \quad \text{and} \quad Q(t+T) = Q(t) \quad \text{all } t$$

and

$$\min_{0 \leqslant t \leqslant T} |P(t) - Q(t)| := \rho > 0.$$

We construct for a potential $U(t, x) = U_P(t, x) + U_Q(t, x)$, where

$$U_P(t, x) := \frac{1}{|x - P(t)|^2} \quad \text{and} \quad U_Q(t, x) := \frac{1}{|x - Q(t)|^2}$$

a function W suitable to show that assumption (1.8) is satisfied.

Consider W_P and $W_Q : \mathbb{R}^3 \backslash \{(t, P(t)) \cup (t, Q(t)) : t \in \mathbb{R}\} \to \mathbb{R}$, defined by

$$W_P(t, x) := \log(\{|x_1 - p_1(t)|^2 + |x_2 - p_2(t)|^2\}^{1/2})$$
$$W_Q(t, x) := \log(\{|x_1 - q_1(t)|^2 + |x_2 - q_2(t)|^2\}^{1/2})$$

and let $W := W_P + W_Q$. W is T-periodic in t and $W \to -\infty$ as $x \to P(t)$ and as $x \to Q(t)$. Moreover:

$$\begin{aligned}
|\nabla W(t, x)|^2 &= \left\{\left(\frac{\partial W_P}{\partial t} + \frac{\partial W_Q}{\partial t}\right)^2 + \left(\frac{\partial W_P}{\partial x_1} + \frac{\partial W_Q}{\partial x_1}\right)^2 + \left(\frac{\partial W_P}{\partial x_2} + \frac{\partial W_Q}{\partial x_2}\right)^2\right\} \\
&= \{|\nabla W_P(t, x)|^2 + |\nabla W_Q(t, x)|^2 + 2\langle \nabla W_P(t, x), \nabla W_Q(t, x)\rangle\} \\
&\leqslant 2\{\{|\nabla W_P(t, x)|^2 + |\nabla W_Q(t, x)|^2\}\} \leqslant 2(U_P(t, x) + U_Q(t, x)) \\
&= 2U(t, x).
\end{aligned}$$

∎

As for the analytical setting, we abbreviate in the following:

$$E := \{x \in H^{1,2}(\mathbb{R}, \mathbb{R}^2) : x(0) = x(T)\} \tag{2.4}$$

with the norm

$$\|x\|^2 := \int_0^T |\dot{x}(t)|^2 \, dt + (|[x]|)^2 \qquad x \in E, \tag{2.5}$$

where

$$[x] := \frac{1}{T}\int_0^T x(t)\,dt \in \mathbb{R}^2. \tag{2.6}$$

Recall that E is compactly embedded in $C([0, T], \mathbb{R}^2)$.

We now set

$$\Lambda := \{x \in E : \text{for all } t,\ 0 \leqslant t \leqslant T : (t, x(t)) \notin S\}. \tag{2.7}$$

Then $\Lambda \subset E$ is open and we define on Λ:

$$I(x) := \int_0^T \left\{\frac{1}{2}|\dot{x}(t)|^2 + U(t, x(t))\right\} dt. \tag{2.8}$$

One shows that $I \in C^1(\Lambda, \mathbb{R})$.

It is convenient for the following to extend I to all of E, by setting $U(t, x) = +\infty$ if $(t, x) \in S$. $I: E \to \mathbb{R} \cup \{+\infty\}$ is (sequentially) weakly lower semicontinuous.

We will establish the existence of critical points of I on Λ which are minima of I over certain subsets of Λ and we will see that these critical points are classical T-periodic solutions. Specifically, for every $k \in \mathbb{Z}^N$, we introduce the set

$$\Lambda_k := \{x \in \Lambda : \deg(x) = k\}, \tag{2.9}$$

so that

$$\Lambda = \bigcup_{k \in \mathbb{Z}^N} \Lambda_k.$$

We will prove that for every $k \in \mathbb{Z}^N$ there is an $x \in \Lambda_k$ s.t.

$$I(x) = \inf_{y \in \Lambda_k} I(y), \qquad I'(x) = 0. \tag{2.10}$$

We proceed in several steps.

To an element $x \in \Lambda$ we can associate the curve x_W:

$$x_W(t) := (t, x(t), W(t, x(t))) \in \mathbb{R}^4 \qquad 0 \leqslant t \leqslant T. \tag{2.11}$$

From the additional assumptions on U one then concludes

LEMMA 1. *If* $x \in \Lambda$, *then*

$$\text{arc length}(x_W) \leqslant T + T^{1/2}(2I(x))^{1/2} + (a_1 I(x) + a_2 T)^{1/2}\,(T + 2I(x))^{1/2}. \tag{2.12}$$

Proof.

$$\begin{aligned}
&\text{arc length}(x_W)\\
&\quad = \int_0^T \left|\frac{d}{dt} x_W(t)\right| dt\\
&\quad \leqslant \int_0^T (1 + |\dot{x}(t)| + |\nabla W(t, x(t))|\,|(1, \dot{x}(t))|)\,dt\\
&\quad \leqslant T + T^{1/2}\left(\int_0^T |\dot{x}|^2\,dt\right)^{1/2} + \left(\int_0^T |\nabla W(t, x(t))|^2\,dt\right)^{1/2}\\
&\qquad \times \left(T + \int_0^T |\dot{x}|^2\,dt\right)^{1/2}\\
&\quad \leqslant T + T^{1/2}(2I(x))^{1/2} + \left(a_1 \int_0^T U(t, x(t))\,dt + a_2 T\right)^{1/2} (T + 2I(x))^{1/2}\\
&\quad \leqslant T + T^{1/2}(2I(x))^{1/2} + (a_1 I(x) + a_2 T)^{1/2}\,(T + 2I(x))^{1/2}. \quad \blacksquare
\end{aligned}$$

Using this lemma we make the following crucial observation:

LEMMA 2. *Take $M > 0$ and set*

$$I^M := \{x \in \Lambda : I(x) \leqslant M\}.$$

Then there is a $\delta = \delta(M) > 0$ such that for $x \in I^M$:

$$|x(t) - P_j(t)| \geqslant \delta, \qquad \textit{all } t,\ 1 \leqslant j \leqslant N.$$

Proof. Assume, by contradiction, that such a δ does not exist. Then there is a $P \in \{P_1, ..., P_N\}$ and a sequence $x_n \in I^M$ satisfying

$$\inf_{0 \leqslant t \leqslant T} |x_n(t) - P(t)| = |x_n(t_n^*) - P(t_n^*)| \to 0 \qquad \text{as} \quad n \to \infty.$$

One of the following two situations holds true:

(a) there is an $\varepsilon > 0$ ($\varepsilon < \rho$), s.t. for a subsequence $x_{j_n} = x_j$:

$$|x_j(t_j) - P(t_j)| = \varepsilon \qquad \text{for all } j \qquad \text{for some} \quad 0 \leqslant t_j \leqslant T;$$

(b) such an ε does not exist and then

$$\|x_n - P\|_{L^\infty} \to 0 \qquad \text{as} \quad n \to \infty.$$

If (a) holds true, since the arc length of a curve connecting two points is larger than the length of the connecting line, we have

$$\text{arc length}(x_{j_W}) \geqslant |W(t_j, x_j(t_j)) - W(t_j^*, x_j(t_j^*))| \to \infty \qquad \text{as} \quad j \to \infty.$$

But this contradicts, in view of Lemma 1, the fact that $I(x_j)$ is bounded.

On the other hand, if (b) holds true, then $\int_0^T U(t, x_n(t))\,dt \to \infty$ as $n \to \infty$; hence $I(x_n)$ is unbounded, a contradiction. ▮

LEMMA 3. *Let* $x \in C(S^1, \mathbb{R}^2 \setminus \{0\})$. *If* $x \approx [x]$, *i.e.,* x *is homotopic in* $\mathbb{R}^2 \setminus \{0\}$ *to the constant loop* $[x]$, *then*

$$\frac{x}{|x|} \approx \text{const.} \qquad \text{on } S^1.$$

Hence

$$\deg \frac{x}{|x|} = 0.$$

Proof. Let $F(s, t)$ be the homotopy which is assumed to exist between x and $[x]$: $F \in C([0, 1] \times S^1, \mathbb{R}^2 \setminus \{0\})$, $F(0, t) = x(t)$, $F(1, t) = [x]$. To simplify notations we call now $e(t) := x(t)/|x(t)|$. Take M big so that the denominator of the following functions f and g does not vanish for $0 \leqslant s \leqslant 1$, and for $t \in S^1$:

$$f(s, t) = \frac{MF(s, t) + se(t)}{|MF(s, t) + se(t)|} \qquad \text{and} \qquad g(s, t) = \frac{M[x] + (1 - s)\,e(t)}{|M[x] + (1 - s)\,e(t)|}.$$

Such an M certainly exists. For example, one can take

$$M = \max \left\{ \frac{2}{\min |F(s, t)|}, \frac{2}{|[x]|} \right\}.$$

Then f gives an homotopy (in $\mathbb{R}^2 \setminus \{0\}$):

$$e(t) \approx \frac{M[x] + e(t)}{|M[x] + e(t)|}$$

and g gives an homotopy (in $\mathbb{R}^2 \setminus \{0\}$):

$$\frac{M[x] + e(t)}{|M[x] + e(t)|} \approx \frac{[x]}{|[x]|}.$$

By the transitivity of the homotopy relation:

$$e(t) = \frac{x(t)}{|x(t)|} \approx \frac{[x]}{|[x]|}. \quad \blacksquare$$

LEMMA 4. *Assume U satisfies* (1.7) *and define the subset $\Lambda^0 := \bigcup_{k \neq 0} \Lambda_k$. Then*

$$\inf_{x \in \Lambda^0} I(x) > 0. \tag{2.13}$$

Proof. We argue by contradiction and assume $\inf\{I(x) : x \in \Lambda^0\} = 0$. Then there is a sequence of elements $x_j \in \Lambda^0$ satisfying

$$I(x_j) \to 0 \qquad \text{as} \quad j \to \infty.$$

Consequently

$$\left\| \frac{dx_j}{dt} \right\|_{L^2} \to 0 \qquad \text{as} \quad j \to \infty \tag{2.14}$$

and

$$\int_0^T U(t, x_j(t))\, dt \to 0 \qquad \text{as} \quad j \to \infty. \tag{2.15}$$

We shall conclude that $\deg(x_j) = 0$ for j large contradicting the assumption $\deg(x_j) \neq 0$. We first conclude from (1.7) and (2.14) that

$$\|x_j\|_{L^\infty} \to \infty \qquad \text{as} \quad j \to \infty.$$

Take now any $P \in \{P_1, ..., P_N\}$ and abbreviate

$$y_j(t) := x_j(t) - P(t).$$

Then

$$\|y_j\|_{L^\infty} \to \infty \qquad \text{as} \quad j \to \infty.$$

Since

$$\|x - [x]\|_{L^\infty} \leqslant T^{1/2}\, \|\dot{x}\|_{L^2} \qquad \text{for every} \quad x \in E, \tag{2.16}$$

we have

$$\|y_j - [y_j]\|_{L^\infty} \leqslant T^{1/2}\, \|\dot{y}_j\|_{L^2} \leqslant T^{1/2}(\|\dot{x}_j\|_{L^2} + \|\dot{P}\|_{L^2}) \leqslant M \qquad \text{all } j$$

for some constant $M > 0$ and we conclude for the mean values that

$$|[y_j]| \to +\infty \qquad \text{as} \quad j \to \infty.$$

Consequently if j is sufficiently large, we have $y_j \approx [y_j]$. Furthermore, by Lemma 3,

$$\deg \frac{y_j}{|y_j|} = 0.$$

This holds true for all $P \in \{P_1, ..., P_N\}$ and consequently $\deg(x_j) = k = 0$ if j is large, a contradiction. ▌

The functional I satisfies on Λ the $(\mathrm{PS})^+$ condition, i.e.,

LEMMA 5. *Assume U satisfies* (1.7) *and* (1.8). *If $a > 0$ and if $\{x_j\} \subset \Lambda$ satisfies*

$$I(x_j) \to a$$

and

$$I'(x_j) \to 0,$$

then $\{x_j\}$ possesses a subsequence which converges to some $x \in \Lambda$.

Proof. (1) We claim that $\{x_j\}$ is bounded in E. Indeed, since $I(x_j)$ is bounded, we conclude that $\|\dot{x}_j\|_{L^2}$ is bounded. Assume now by contradiction that $\{[x_j]\} \subset \mathbb{R}^2$ is not bounded. Then there is a subsequence (which we will continue to denote x_j) satisfying:

$$|[x_j]| \to \infty \qquad \text{as} \quad j \to \infty.$$

From

$$\|x_j - [x_j]\|_{L^\infty} \leqslant T^{1/2} \|\dot{x}_j\|_{L^2}$$

we conclude that

$$\inf_{0 \leqslant t \leqslant T} |x_j(t)| \to \infty \qquad \text{as} \quad j \to \infty,$$

and consequently

$$\int_0^T U(t, x_j(t))\, dt \to 0 \qquad \text{as} \quad j \to \infty.$$

Moreover

$$\int_0^T \frac{\partial}{\partial x} U(t, x_j(t))(x_j(t) - [x_j])\, dt \to 0 \qquad \text{as} \quad j \to \infty.$$

Observe that

$$I(x_j)=\frac{1}{2}I'(x_j)(x_j-[x_j])+\int_0^T U(t,x_j)\,dt-\frac{1}{2}\int_0^T \frac{\partial}{\partial x}U(t,x_j)(x_j-[x_j])\,dt.$$

Since $x_j-[x_j]$ is bounded in L^∞, also the first term on the right hand side tends to zero; therefore $I(x_j)\to 0$ (as $j\to\infty$) contradicting the assumption $I(x_j)\to a>0$.

(2) Since $\{x_j\}$ is bounded in E, there is a subsequence such that

$$\begin{aligned} x_j &\to x \qquad \text{in } E \text{ weakly} \\ x_j &\to x \qquad \text{in } C([0,T],\mathbb{R}^2). \end{aligned}$$

Since $I(x_j)$ is bounded, we conclude by Lemma 2 that $x\in\Lambda$.

(3) We claim that $x_j\to x$ in E. A computation shows that

$$\begin{aligned} \|x_j-x\|_E^2 &= \int_0^T |\dot{x}_j-\dot{x}|^2\,dt+[x_j-x]^2 \\ &= -\int_0^T (\dot{x}_j-\dot{x})\dot{x}\,dt-[x_j-x][x] \\ &\quad+\int_0^T (\dot{x}_j-\dot{x})\dot{x}_j\,dt+[x_j-x][x_j]. \end{aligned}$$

Therefore it only remains to show that

$$\int_0^T (\dot{x}_j-\dot{x})\dot{x}_j\,dt\to 0$$

as $j\to\infty$. But

$$\int_0^T \dot{x}_j(\dot{x}_j-\dot{x})\,dt=I'(x_j)(x_j-x)-\int_0^T \frac{\partial}{\partial x}U(t,x_j)(x_j-x)\,dt.$$

Since x_j-x is bounded in L^∞, and since $(\partial/\partial x)\,U(t,x_j)$ is bounded in t uniformly in j, the claim follows. ∎

Finally we recall the "deformation lemma" in a form which is adapted to the problem. Recall that for $a\in\mathbb{R}$, we use the notation

$$I^a:=\{x\in\Lambda : I(x)\leqslant a\}.$$

LEMMA 6. *Assume U satisfies* (1.7) *and* (1.8) *and I is defined as in* (2.7).

If $c > 0$ is not a critical value of I, then for every $\bar{\varepsilon} > 0$ there exists an $\varepsilon > 0$ and a map $\eta \in C([0, 1] \times \Lambda, \Lambda)$ s.t.

(1) $\eta(0, x) = x$ *all* $x \in \Lambda$,

(2) $\eta(s, \cdot) \equiv \eta_s(\cdot)$ *is a homeomorphism of* Λ *in* Λ *for* $0 \leqslant s \leqslant 1$,

(3) $\eta(s, x) = x$ *if* $I(x) \notin (c - \bar{\varepsilon}, c + \bar{\varepsilon})$,

(4) $I(\eta(s, x)) \leqslant I(x)$ *all* $s \geqslant 0$,

(5) $\eta(1, I^{c+\varepsilon}) \subset I^{c-\varepsilon}$,

(6) $\eta(s, \cdot): \Lambda_k \to \Lambda_k$ $\forall k \in \mathbb{Z}^N$.

Proof. By Lemma 5, I satisfies the $(PS)^+$ condition so that the points (1)–(5) in this statement are well known. More precisely it is known that to a functional $I \in C^1(H, \mathbb{R})$ (H Hilbert space) a map η is associated s.t. $\eta \in C([0, 1] \times H, H)$ and η satisfies (1)–(5) with Λ replaced by H.

For a detailed proof we refer to [32]. Here we only have to prove that (with our assumptions) $\eta(s, \cdot): \Lambda \to \Lambda$ and that (6) holds true. We recall that the map η in the statement (usually called the "steepest descent flow") is the solution of a differential equation of the form

$$\frac{d\eta}{ds} = \omega(\eta)\, \psi(\eta) \qquad \eta(0, x) = x,$$

where $0 \leqslant \omega \leqslant 1$ is a cut-off function and ψ is a pseudogradient vectorfield for $-I'$.

The invariance of Λ under the map η_s follows by Lemma 2.1 and point (4). Take indeed an element $x \in \Lambda$. Obviously $I(x) < +\infty$. If $I'(x) = 0$, it is $\eta(s, x) = x \in \Lambda$ for $0 \leqslant s \leqslant 1$. If $I'(x) \neq 0$, letting the steepest descent flow evolve, we obtain (by (4)):

$$I(\eta(s, x)) \leqslant I(x) < +\infty \qquad \text{for} \quad 0 \leqslant s \leqslant 1.$$

But then Lemma 2.1 implies that $\eta(s, x) \in \Lambda$ for $0 \leqslant s \leqslant 1$. To prove (6), we just observe that

$$\frac{\eta_s(x)(t) - P_j(t)}{|\eta_s(x)(t) - P_j(t)|}$$

is for every $j = 1, ..., N$ a continuous function from $[0, 1] \times S^1$ in S^1, which establishes for $0 \leqslant s \leqslant 1$ an homotopy (in $\mathbb{R}^2 \backslash \{0\}$) between

$$\frac{x(t) - P_j(t)}{|x(t) - P_j(t)|} \qquad \text{and} \qquad \frac{\eta_s(x)(t) - P_j(t)}{|\eta_s(x)(t) - P_j(t)|}.$$

From the homotopy-invariance of the degree: $\deg(\eta(s, x)) = \deg(x)$. Hence: $x \in \Lambda_k$ implies $\eta(s, x) \in \Lambda_k$. ∎

LEMMA 7. *Assume* $\{x_j\} \subset \Lambda_k$ *satisfies*

$$x_j \to x \in \Lambda \qquad \textit{in } C([0, T], \mathbb{R}^2).$$

Then $x \in \Lambda_k$.

Proof. By assumption

$$\sup_{0 \leqslant t \leqslant T} |x_j(t) - x(t)| \to 0 \qquad \text{as} \quad j \to \infty.$$

Hence for every $\varepsilon > 0$ there exists a $j_\varepsilon \in \mathbb{N}$ s.t. if $j \geqslant j_\varepsilon$, then

$$|x_j(t) - x(t)| \leqslant \varepsilon \qquad 0 \leqslant t \leqslant T. \tag{2.17}$$

If r denotes

$$r = \min_{P \in \{P_1, ..., P_N\}} \{ \min_{0 \leqslant t \leqslant T} |x(t) - P(t)| \}$$

we will consider a fixed ε: $\varepsilon < r/3$ and $j \geqslant j_\varepsilon$. It is then

$$\begin{aligned} |x_j(t) - P(t)| &\geqslant |x(t) - P(t)| - |x_j(t) - x(t)| \\ &\geqslant r - \frac{r}{3} = \frac{2}{3} r \qquad \text{for every} \quad P \in \{P_1, ..., P_N\},\ 0 \leqslant t \leqslant T. \end{aligned}$$

Call $\delta_j(t) := x(t) - x_j(t)$. By (2.17), $|\delta_j(t)| < r/3$ $0 \leqslant t \leqslant T$. Consider for $0 \leqslant s \leqslant 1$, $0 \leqslant t \leqslant T$:

$$F_{P,j}(s, t) = \frac{x_j(t) - P(t) + s\,\delta_j(t)}{|x_j(t) - P(t) + s\,\delta_j(t)|}. \tag{2.18}$$

The denominator of (2.18) never vanishes, and for every $P \in \{P_1, ..., P_N\}$, for every fixed $j \geqslant j_\varepsilon$, $F_{P,j}(s, t)$ is continuous and gives a homotopy between

$$\frac{x_j(t) - P(t)}{|x_j(t) - P(t)|} \qquad \text{and} \qquad \frac{x(t) - P(t)}{|x(t) - P(t)|}.$$

Hence for every $P \in \{P_1, ..., P_N\}$ and $j \geqslant j_\varepsilon$ these two functions have the same degree and this amounts to saying that $x \in \Lambda_k$. ∎

We can now give the

Proof of Theorem 1. Assume $k \neq 0$, and define

$$c := \inf_{x \in \Lambda_k} I(x). \tag{2.19}$$

By Lemma 4, $c > 0$.

There is a minimizing sequence $\{x_j\} \subset \Lambda_k$ satisfying

$$I(x_j) \to c \qquad \text{and} \qquad I'(x_j) \to 0. \tag{2.20}$$

Indeed if there is not such sequence, then c is not a critical value and consequently, by Lemma 6,

$$\eta(1, I^{c+\varepsilon} \cap \Lambda_k) \subset I^{c-\varepsilon} \cap \Lambda_k,$$

for some $\varepsilon > 0$, contradicting the definition of c. This proves the above claim. By Lemma 5 (the $(PS)^+$ condition) we conclude a subsequence still denoted $\{x_j\} \subset \Lambda_k$, such that $x_j \to x \in \Lambda$ in E and, moreover,

$$c = \lim_{j \to \infty} I(x_j) = I(x) \tag{2.21}$$

$$0 = \lim_{j \to \infty} I'(x_j) = I'(x). \tag{2.22}$$

By Lemma 7, $x \in \Lambda_k$.

It remains to prove that $x \in C^2([0, T], \mathbb{R}^2)$. This follows from the next lemma.

LEMMA 8. *Assume U satisfies* (1.7) *and* (1.8) *and let $x \in \Lambda$ be a critical point, $I'(x) = 0$. Then $x \in C^2([0, T], \mathbb{R}^2)$,*

$$x(0) = x(T),$$

$$\dot{x}(0) = \dot{x}(T)$$

and

$$\ddot{x} = \frac{\partial}{\partial x} U(t, x).$$

Proof. If $x \in \Lambda \subset E$, then x is continuous and so is $(\partial/\partial x)\, U(t, x(t))$. Write

$$\frac{\partial}{\partial x} U(t, x(t)) = \frac{d}{dt} \int_0^t \frac{\partial}{\partial x} U(s, x(s))\, ds$$

and consider the scalar product with $y \in E$, $y(0)=0$. Integrate on $[0, T]$:

$$\int_0^T \left\langle \frac{\partial}{\partial x} U(t, x), y \right\rangle dt = -\int_0^T \left\langle \int_0^t \frac{\partial}{\partial x} U(s, x)\, ds, \dot{y} \right\rangle dt.$$

Since

$$\begin{aligned} &\int_0^T \left\{ \langle \dot{x}, \dot{y} \rangle + \left\langle \frac{\partial}{\partial x} U(t, x), y \right\rangle \right\} dt = 0 \qquad \text{for every} \quad y \in E, \\ &\int_0^T < \dot{x} - \int_0^t \frac{\partial}{\partial x} U(s, x)\, ds, \dot{y} > dt = 0 \end{aligned} \tag{2.23}$$

holds true for every $y \in E$, s.t. $y(0)=0$. Hence

$$\dot{x} - \int_0^t \frac{\partial}{\partial x} U(s, x)\, ds = \text{const.} \qquad \text{a.e.} \tag{2.24}$$

Moreover, since $x \in \Lambda$, it can be seen that $x \in H^{2,2}$ and therefore (via Sobolev embedding) $\dot{x} \in C$. By this and (2.24) $\dot{x} \in C^1$.

We can now integrate (2.23) by parts to get

$$\begin{aligned} &\langle \dot{x}(T), y(T) \rangle - \langle \dot{x}(0), y(0) \rangle \\ &\quad = \int_0^T \langle \ddot{x}(t), y(t) \rangle\, dt - \int_0^T \left\langle \frac{\partial}{\partial x} U(t, x(t)), y(t) \right\rangle dt \qquad \text{for every} \quad y \in E. \end{aligned}$$

The rhs is equal to zero, hence $\dot{x}(T) = \dot{x}(0)$ follows. ∎

3. Multiplicity of Subharmonic Solutions

The argument above leads immediately also to the existence of infinitely many subharmonic solutions (namely solutions of period nT, $n \in \mathbb{N}^+$) with minimal period for systems of the form (1.3). Indeed, since for every $n \in \mathbb{N}^+$: $U(t + nT, x) = U(t, x)$ for all $(t, x) \in \mathbb{R}^3 \setminus S$, the forced system

$$\ddot{x} = \frac{\partial}{\partial x} U(t, x) \tag{1.3}$$

admits in particular nT as a period.

For every $n \in \mathbb{N}^+$, Theorem 1 guarantees the existence of infinitely many nT-periodic solutions. In particular it guarantees the existence of ∞^{N-1} nT-periodic solutions x^j, $1 \leqslant j \leqslant N$, with

$$\deg(x^j) = (k_1, ..., k_N) \in \mathbb{Z}^N.$$

where $k_j = +1$ and ∞^{N-1} nT-periodic solutions y^j, $1 \leqslant j \leqslant N$, with

$$\deg(y^j) = (k_1, ..., k_N) \in \mathbb{Z}^N, \qquad \text{where} \quad k_j = -1.$$

We claim that the period nT of the x^j's and the y^j's is minimal.

We prove the claim for a solution $x^j := x$ (the proof for any y^j is identical): We know that $x(t+nT) = x(t)$ and

$$\deg(x_{[0,nT]}) = (\bar{k}_1, ..., \bar{k}_{j-1}, 1, \bar{k}_{j+1}, ..., \bar{k}_N) \in \mathbb{Z}^N. \tag{3.1}$$

Suppose there exists an $m \in \mathbb{Z}^+$, such that

$$x\left(t + \frac{n}{m} T\right) = x(t).$$

Looking at x as a function of period $(n/m)T$, one obtains

$$\deg(x_{[0,nT/m]}) = k, \tag{3.2}$$

for a certain vector $k = (k_1, ..., k_{j-1}, k_j, k_{j+1}, ..., k_N) \in \mathbb{Z}^N$. Consequently, when one looks at x as at a function of period nT, the degree has to be

$$\deg(x_{[0,nT]}) = (mk_1, ..., mk_{j-1}, mk_j, mk_{j+1}, ..., mk_N) \in \mathbb{Z}^N, \tag{3.3}$$

$(k_1, ..., k_{j-1}, k_j, k_{j+1}, ..., k_N)$ as in (3.2). Comparing (3.1) and (3.3) one deduces

$$1 = mk_j \qquad m \in \mathbb{Z}^+,\ k_j \in \mathbb{Z}. \tag{3.4}$$

And the only possibility for (3.4) to hold true is $m = 1$, $k_j = 1$.

Summarizing, we have the following corollary of Theorem 1:

THEOREM 4. *Assume that U satisfies* (1.7) *and* (1.8). *Then the system*

$$\ddot{x} = \frac{\partial}{\partial x} U(t, x) \qquad (t, x) \in \mathbb{R}^3 \backslash S,$$

admits for every $n \in \mathbb{Z}^+$ infinitely many distinct nT-periodic solutions with minimal period.

4. Generalized Solutions for the Restricted Three Body Problem

In this section we prove Theorem 3. We proceed by approximation. Recall that

$$\rho := \min_{\substack{0 \leqslant t \leqslant T \\ 1 \leqslant k \neq j \leqslant N}} |P_k(t) - P_j(t)| > 0. \tag{4.1}$$

Set

$$D := \max_{j=1,\dots,N} \|P_j'(t)\|_{L^\infty}. \tag{4.2}$$

LEMMA 9. *Assume $0 < \varepsilon < \frac{2}{5}\rho$. Then there exists $U_\varepsilon \in C^2(\mathbb{R}^3 \setminus S, \mathbb{R})$, such that*

$$\begin{aligned}
&(1) \quad U_\varepsilon(t, x) = U(t, x) \qquad \text{if} \quad |x - P_j(t)| \geqslant \varepsilon,\ 1 \leqslant j \leqslant N \\
&(2) \quad U_\varepsilon(t, x) \geqslant U(t, x) \qquad \text{all} \quad (t, x) \in \mathbb{R}^3 \setminus S \\
&(3) \quad U_\varepsilon \text{ meets } (1.7) \text{ and } (1.8);
\end{aligned} \tag{4.3}$$

moreover the constants a_1 and a_2 appearing in (1.8) *are independent of ε. They are given below.*

Proof. For $0 < \varepsilon < \frac{2}{5}\rho$ and for $1 \leqslant j \leqslant N$ let $\psi_{\varepsilon,j} \in C^2(\mathbb{R}^3, \mathbb{R}^+)$ be such that

$$\psi_{\varepsilon,j}(t, x)\ (= \tilde{\psi}_\varepsilon(|x - P_j(t)|)) := \begin{cases} 1 & \text{if} \quad |x - P_j(t)| \leqslant \dfrac{\varepsilon}{2} \\ 0 & \text{if} \quad |x - P_j(t)| \geqslant \varepsilon \end{cases} \tag{4.4}$$

and

$$|\nabla \psi_{\varepsilon,j}(t, x)|^2 \leqslant \frac{C}{\varepsilon^2}, \tag{4.5}$$

C being a positive constant. It is easy to see that such a function exists. Set

$$U_\varepsilon(t, x) := U(t, x) + \frac{\varepsilon^2}{\left|\log \dfrac{\varepsilon}{2}\right|^2} \sum_{j=1}^{N} \psi_{\varepsilon,j}(t, x) \frac{1}{|x - P_j(t)|^2} \tag{4.6}$$

$$W_\varepsilon(t, x) := \frac{\varepsilon}{\left|\log \dfrac{\varepsilon}{2}\right|} \sum_{j=1}^{N} \psi_{\varepsilon,j}(t, x) \log |x - P_j(t)|. \tag{4.7}$$

We have to prove that

$$|\nabla W_\varepsilon(t, x)|^2 \leqslant a_1 U_\varepsilon(t, x) + a_2 \qquad \text{for} \quad (t, x) \in \mathbb{R}^3 \setminus S \tag{4.8}$$

with two positive constants a_1, a_2.

It is of course sufficient to prove (4.8) for $0 \leqslant t \leqslant T$ rather than for $t \in \mathbb{R}$.

We observe that $[0, T] \times \mathbb{R}^2$ is the union of three regions:

$$[0, T] \times \mathbb{R}^2 = X_1 \cup X_2 \cup X_3$$

where (denoting $B_r(y) = \{x : |x - y| < r\}$):

$$X_1 = \bigcup_{j=1}^{N} \{(t, x) : x \in B_{\varepsilon/2}(P_j(t)) : 0 \leqslant t \leqslant T\},$$

$$X_2 = \bigcup_{j=1}^{N} \left\{(t, x) : \frac{\varepsilon}{2} \leqslant |x - P_j(t)| \leqslant \varepsilon : 0 \leqslant t \leqslant T\right\},$$

$$X_3 = \mathbf{C} \bigcup_{j=1}^{N} \{(t, x) : x \in \overline{B_\varepsilon(P_j(t))} : 0 \leqslant t \leqslant T\}.$$

($\mathbf{C}\ X$ denotes the complement of X in $[0, T] \times \mathbb{R}^2$). Hence: $(\mathbb{R}^3 \backslash S) \cap ([0, T]) \times \mathbb{R}^2) = (X_1 \backslash (X_1 \cap S)) \cup X_2 \cup X_3$. If $(t, x) \in X_3$, $W_\varepsilon(t, x) = 0$ and therefore $|\nabla W_\varepsilon(t, x)|^2 = 0$. If $(t, x) \in X_1 \backslash (X_1 \cap S)$, then according to definitions (4.6) and (4.7),

$$U_\varepsilon(t, x) := U(t, x) + \frac{\varepsilon^2}{\left|\log \frac{\varepsilon}{2}\right|^2} \frac{1}{|x - P(t)|^2}$$

and

$$W_\varepsilon(t, x) := \frac{\varepsilon}{\left|\log \frac{\varepsilon}{2}\right|} \log |x - P(t)|$$

for a certain $P \in \{P_1, ..., P_N\}$. A simple computation shows

$$|\nabla W_\varepsilon(t, x)|^2 \leqslant \frac{\varepsilon^2}{\left|\log \frac{\varepsilon}{2}\right|^2} \frac{2D^2 + 1}{|x - P(t)|^2} \leqslant (2D^2 + 1)\, U_\varepsilon(t, x), \qquad (D \text{ as in } (4.2)).$$

If $(t, x) \in X_2$, then

$$U_\varepsilon(t, x) = U(t, x) + \frac{\varepsilon^2}{\left|\log \frac{\varepsilon}{2}\right|^2} \psi_\varepsilon(t, x) \frac{1}{|x - P(t)|^2}$$

$$W_\varepsilon(t, x) = \frac{\varepsilon}{\left|\log \frac{\varepsilon}{2}\right|} \psi_\varepsilon(t, x) \log |x - P(t)|$$

for a certain $P = P_j \in \{P_1, ..., P_N\}$ and the corresponding $\psi_\varepsilon = \psi_{\varepsilon, j}$. Then

$$\begin{aligned}
|\nabla W_\varepsilon(t, x)|^2 &= \frac{\varepsilon^2}{\left|\log \frac{\varepsilon}{2}\right|^2} |(\log |x - P(t)|)(\nabla \psi_\varepsilon(t, x)) \\
&\quad + (\psi_\varepsilon(t, x))(\nabla \log |x - P(t)|)|^2 \\
&\leqslant \frac{2\varepsilon^2}{\left|\log \frac{\varepsilon}{2}\right|^2} |(\log |x - P(t)|)^2 |\nabla \psi_\varepsilon|^2 \\
&\quad + (\psi_\varepsilon)^2 |\nabla \log |x - P(t)||^2|,
\end{aligned}$$

since by (4.4) and (4.5): $|\nabla \psi_\varepsilon|^2 \leqslant C/\varepsilon^2$, $|\psi_\varepsilon|^2 \leqslant |\psi_\varepsilon| = \psi_\varepsilon$ and, if $\varepsilon/2 \leqslant |x - P(t)| \leqslant \varepsilon < 1$, then

$$\begin{aligned}
|\log \varepsilon|^2 \leqslant (\log |x - P(t)|)^2 \leqslant \left(\left|\log \frac{\varepsilon}{2}\right|^2\right) \\
&\leqslant \frac{2\varepsilon^2}{\left|\log \frac{\varepsilon}{2}\right|^2} \left\{\frac{C}{\varepsilon^2}\left|\log \frac{\varepsilon}{2}\right|^2 + \psi_\varepsilon \frac{(2D^2+1)}{|x - P(t)|^2}\right\} \\
&= 2C + 2\frac{(2D^2+1)}{|x - P(t)|^2} \psi_\varepsilon \frac{\varepsilon^2}{\left|\log \frac{\varepsilon}{2}\right|^2} \\
&\leqslant 2C + 2(2D^2 + 1)\, U_\varepsilon(t, x), \qquad (C \text{ as in } (4.5),\ D \text{ as in } (4.2)).
\end{aligned}$$

Summarizing: for all $(t, x) \in \mathbb{R}^3 \backslash S$, for all ε: $0 < \varepsilon < \frac{2}{3}\rho$,

$$|\nabla W_\varepsilon(t, x)|^2 \leqslant 2(2D^2 + 1)\, U_\varepsilon(t, x) + 2C = a_1 U_\varepsilon(t, x) + a_2,$$

where $a_1 = 2(2D^2 + 1)$ and $a_2 = 2C$ (C as in (4.5), D as in (4.2)). ∎

Define now the family of functionals on Λ:

$$I_\varepsilon(x) := \int_0^T \left\{\frac{1}{2} |\dot{x}(t)|^2 + U_\varepsilon(t, x(t))\right\} dt. \tag{4.9}$$

Fix in the following $k \in \mathbb{Z}^N$, $k \neq 0$.

Take, using Theorem 1, the critical points $x_\varepsilon \in \Lambda_k$, satisfying

$$I_\varepsilon(x_\varepsilon) := \inf_{x \in \Lambda_k} I_\varepsilon(x), \qquad I'_\varepsilon(x_\varepsilon) = 0. \tag{4.10}$$

LEMMA 10. *There exist two constants* $0<\alpha<\beta$ *independent of* ε, *such that*

$$\alpha \leqslant I_\varepsilon(x_\varepsilon) \leqslant \beta \tag{4.11}$$

for all $\varepsilon>0$ *sufficiently small.*

In order to prove Lemma 10, we need some previous work. We point out that the estimate (4.15), we are next going to prove, is related to an estimate derived by Bahri and Rabinowitz in a similar situation in [22].

We associate to the potential U a function φ: $\mathbb{R}^+ \to \mathbb{R}$ defined by

$$\varphi(r) := \sup_{\substack{0 \leqslant t \leqslant T \\ 0 \leqslant |x| \leqslant r}} ((U(t,x))^{-1}), \tag{4.12}$$

where we set $(U(t,x))^{-1}=0$ if $(t,x)\in S$.

In view of (1.7) the extended (on $\mathbb{R}^3$) function $(U(t,x))^{-1}$ is continuous. Consequently the sup appearing in (4.12) is actually a max. φ is a continuous function, monotone increasing and $\varphi(r)\to\infty$ as $r\to\infty$ (in view of (1.7)(ii)).

We also set

$$M := \max_{j=1,\dots,N} \|P_j(t)\|_{L^\infty} \tag{4.13}$$

(recall assumption (1.4)).

LEMMA 11. *If U satisfies* (1.7) *and for* $k\in\mathbb{Z}^N$, $k\neq 0$,

$$c = \inf_{x\in\Lambda_k} I(x), \tag{4.14}$$

then

$$T \leqslant c\varphi(2(2Tc)^{1/2}+2M). \tag{4.15}$$

Proof. For $\theta>0$ set

$$\varphi^{(\theta)}(r) = \theta r + \varphi(r). \tag{4.16}$$

In the first part of this proof we will consider θ fixed and we will abbreviate:

$$\psi(r) = \varphi^{(\theta)}(r). \tag{4.17}$$

ψ has the same properties of φ and it is strictly monotone increasing.

Let x be in Λ_k. We can split $x=[x]+X$, where X is orthogonal in E to $\mathbb{R}^2$. Suppose

$$I(x)\leqslant d \qquad \text{for a } d>0. \tag{4.18}$$

Then by (4.18), (2.8), and (2.16)

$$\|X\|_{L^\infty}\leqslant (2Td)^{1/2}. \tag{4.19}$$

In view of (4.12), (4.16), (4.17), (4.19), and the Hölder inequality,

$$\begin{aligned} T&=\int_0^T dt=\int_0^T \left(\frac{U(t,x(t))}{U(t,x(t))}\right)^{1/2} dt \\ &\leqslant\left(\int_0^T U(t,x(t))\,dt\right)^{1/2}\left(\int_0^T (U(t,x(t)))^{-1}\,dt\right)^{1/2} \\ &\leqslant d^{1/2}\left(\int_0^T \psi(|x(t)|)\,dt\right)^{1/2}. \end{aligned} \tag{4.20}$$

Substituting $x=[x]+X$ into (4.20) and using (4.19) and the monotonicity of ψ, we have

$$\frac{T^2}{d}\leqslant T\psi(|[x]|+\|X\|_{L^\infty})\leqslant T\psi(|[x]|+(2Td)^{1/2}). \tag{4.21}$$

Consequently

$$\psi^{-1}\left(\frac{T}{d}\right)-(2Td)^{1/2}\leqslant |[x]| \tag{4.22}$$

and by (4.19) and (4.22) for $0\leqslant s\leqslant 1$:

$$|[x]+sX(t)|\geqslant |[x]|-\|X\|_{L^\infty}\geqslant \psi^{-1}\left(\frac{T}{d}\right)-2(2Td)^{1/2}. \tag{4.23}$$

Consider now for $1\leqslant j\leqslant N$ the loop

$$x(t)-P_j(t) \qquad 0\leqslant t\leqslant T$$

and evaluate, for $0\leqslant s\leqslant 1$,

$$\begin{aligned} &|[x(t)-P_j(t)]+s((x(t)-P_j(t))-[x(t)-P_j(t)])| \\ &\quad=|[x]+s(x(t)-[x])+[-P_j]+s(-P_j(t)-[-P_j])| \\ &\quad\geqslant |[x]+sX(t)|-|[-P_j]+s(-P_j(t)-[-P_j])| \\ &\quad\geqslant \psi^{-1}\left(\frac{T}{d}\right)-2(2Td)^{1/2}-2\,\|P_j(t)]\|_{L^\infty}, \end{aligned} \tag{4.24}$$

where the last inequality follows by (4.23). Suppose that for every j: $1 \leqslant j \leqslant N$

$$\psi^{-1}\left(\frac{T}{d}\right) - 2(2Td)^{1/2} - 2\,\|P_j(t)]\|_{L^\infty} > 0. \tag{4.25}$$

If this is the case, (4.24) implies that $x(t) - P_j(t)$ is homotopic (in $\mathbb{R}^2 \setminus \{0\}$) to its mean value $[x(t) - P_j(t)]$ for all $1 \leqslant j \leqslant N$. But then Lemma 3 gives

$$\deg \frac{x(t) - P_j(t)}{|x(t) - P_j(t)|} = 0 \qquad \text{for every} \quad j = 1, ..., N.$$

Now, if we take d in (4.18) to be $d = c + \varepsilon$, with c as in (4.14) and any $\varepsilon > 0$, the definition of inf implies the existence of at least one element x in Λ_k, s.t. $I(x) \leqslant c + \varepsilon$. Since the argument above can be carried out for every x in Λ_k satisfying (4.18), it is evident that assuming (4.25) takes to a contradiction (we are assuming $k \neq 0$). Consequently (4.25) cannot hold true for every $j = 1, ..., N$ if $d = c + \varepsilon$, $\varepsilon > 0$.

Equivalently: for $d = c + \varepsilon$, $\varepsilon > 0$, there is at least one $P \in \{P_1, ..., P_N\}$, such that

$$\psi^{-1}\left(\frac{T}{d}\right) - 2(2Td)^{1/2} - 2\,\|P(t)]\|_{L^\infty} \leqslant 0,$$

and then also

$$\frac{T}{d} \leqslant \psi(2(2Td)^{1/2} + 2\,\|P(t)]\|_{L^\infty}). \tag{4.26}$$

Since this is the case for every $\varepsilon > 0$, the inequality (4.26) holds true also with d replaced by c:

$$T \leqslant c\psi(2(2Tc)^{1/2} + 2\,\|P(t)]\|_{L^\infty}).$$

Recall (4.16) and (4.17): $\psi(r) = \varphi^{(\theta)}(r) = \theta r + \varphi(r)$. Letting $\theta \to 0$, one has:

$$T \leqslant c\varphi(2(2Tc)^{1/2} + 2\,\|P(t)]\|_{L^\infty}),$$

and in view of the monotonicity of φ (since $\|P(t)]\|_{L^\infty} \leqslant M$), (4.15) follows. ∎

Lemma 11 implies the following relation between the critical values obtained with Theorem 1 and the period T of the corresponding solutions. Since for $\varepsilon \in (0, \frac{2}{5}\rho)$ U_ε satisfies (1.7), denoting

$$c_\varepsilon := I_\varepsilon(x_\varepsilon) = \inf_{x \in \Lambda_k} I_\varepsilon(x) \tag{4.27}$$

and

$$\varphi_\varepsilon(r) := \sup_{\substack{0 \leqslant t \leqslant T \\ 0 \leqslant |x| \leqslant r}} ((U_\varepsilon(t, x))^{-1}), \tag{4.28}$$

(where $(U_\varepsilon(t, x))^{-1} := 0$ if $(t, x) \in S$), one obtains:

$$T \leqslant c_\varepsilon \varphi_\varepsilon(2(2Tc_\varepsilon)^{1/2} + 2M). \tag{4.29}$$

We are now ready to give the

Proof of Lemma 10. Take a loop $y \in \Lambda_k$. Then

$$|y(t) - P_j(t)| \geqslant \sigma > 0 \qquad 0 \leqslant t \leqslant T,\ 1 \leqslant j \leqslant N$$

for a certain positive σ.

We will consider from now on $\varepsilon \in (0, \bar{\varepsilon})$, where $\bar{\varepsilon} := \min(\frac{2}{5}\rho, \sigma)$. For $\varepsilon \in (0, \bar{\varepsilon})$ it is, in view of (4.20),

$$c_\varepsilon := I_\varepsilon(x_\varepsilon) \leqslant I_\varepsilon(y) = I(y) := \beta. \tag{4.30}$$

In order to get a lower bound for $I_\varepsilon(x_\varepsilon)$, we observe that, by (4.12), (4.28), and (4.3)(2),

$$\varphi_\varepsilon(r) \leqslant \varphi(r) \qquad \text{for} \quad r \geqslant 0. \tag{4.31}$$

Let

$$\alpha := \inf_{0 < \varepsilon < \bar{\varepsilon}} c_\varepsilon. \tag{4.32}$$

By definition

$$\alpha \leqslant I_\varepsilon(x_\varepsilon).$$

We prove now that $\alpha > 0$.

By (4.31) and (4.29) one gets

$$T \leqslant c_\varepsilon \varphi(2(2Tc_\varepsilon)^{1/2} + 2M).$$

Passing to the inf (over ε):

$$T \leqslant \alpha\varphi(2(2T\alpha)^{1/2} + 2M),$$

and this last inequality implies $\alpha > 0$. ∎

LEMMA 12. *There is a sequence $\varepsilon_j \to 0$ and an $x \in E$, such that*

$$x_{\varepsilon_j} \to x \qquad \textit{weakly in } E \tag{4.33}$$

$$x_{\varepsilon_j} \to x \qquad \textit{in } C([0, T], \mathbb{R}^2). \tag{4.34}$$

Proof. We prove that there is a constant $\gamma > 0$ independent of ε, such that for $\varepsilon \in (0, \bar{\varepsilon})$

$$\|x_\varepsilon\|_E \leqslant \gamma.$$

As a consequence of Lemma 10 one has

$$\|\dot{x}_\varepsilon\|_{L^2}^2 \leqslant 2\beta \tag{4.35}$$

and

$$\int_0^T U_\varepsilon(t, x_\varepsilon(t))\, dt \leqslant \beta. \tag{4.36}$$

We show here that also $|[x_\varepsilon]|$ is bounded independently on ε.

Assume by contradiction this is not the case. Then there is a subsequence x_{ε_j} (we will abbreviate $x_{\varepsilon_j} = x_j$, $U_{\varepsilon_j} = U_j$, and $I_{\varepsilon_j} = I_j$; $\varepsilon_j \to 0$ as $j \to \infty$) satisfying

$$|[x_j]| \to \infty \qquad \text{as} \quad j \to \infty.$$

From

$$\|x - [x]\|_{L^\infty} \leqslant T^{1/2}\, \|\dot{x}\|_{L^2} \qquad x \in E,$$

we conclude that

$$\inf_t |x_j(t)| \to \infty \qquad \text{as} \quad j \to \infty$$

and

$$\|x_j - [x_j]\|_{L^\infty} \leqslant (2T\beta)^{1/2}.$$

Consequently (by (4.3)(3))

$$\int_0^T U_j(t, x_j(t))\, dt \to 0 \qquad \text{as} \quad j \to +\infty$$

and

$$\int_0^T \frac{\partial}{\partial x} U_j(t, x_j(t))(x_j(t) - [x_j])\, dt \to 0 \qquad \text{as} \quad j \to +\infty.$$

Therefore, since

$$c_{\varepsilon_j} = I_j(x_j) = \frac{1}{2} I_j'(x_j)(x_j - [x_j])$$
$$-\frac{1}{2} \int_0^T \frac{\partial}{\partial x} U_j(t, x_j)(x_j - [x_j])\, dt + \int_0^T U_j(t, x_j)\, dt,$$

it follows that

$$c_{\varepsilon_j} \to 0 \qquad \text{as} \quad j \to \infty.$$

But this is in contradiction to

$$\inf_{0<\varepsilon<\bar{\varepsilon}} c_\varepsilon = \alpha > 0.$$

The statement of the Lemma is now immediate. ∎

Remark. Even if every loop x_j is bounded away from the singularity ($x_j \in \Lambda$), we are not able to deduce the same property for the limiting loop x. What can be said is that the set of the times s.t. $(t, x(t)) \in S$ is a set of measure zero. Indeed, for all ε sufficiently small:

$$\int_0^T U_\varepsilon(t, x_\varepsilon(t))\, dt \leqslant \beta.$$

Let $\delta > 0$ and define

$$\chi_\delta(s) = 0 \qquad \text{for} \quad s < \delta$$

$$\chi_\delta(s) = 1 \qquad \text{for} \quad s \geqslant \delta$$

By (4.36):

$$\int_0^T \chi_\delta(|x_\varepsilon(t) - P_\mu(t)|)\, \chi_\delta(|x_\varepsilon(t) - P_\nu(t)|)\, U_\varepsilon(t, x_\varepsilon(t))\, dt \leqslant \beta,$$

and, in view of the uniform convergence of the x_ε's, this gives

$$\int_0^T \chi_\delta(|x(t) - P_\mu(t)|)\, \chi_\delta(|x(t) - P_\nu(t)|)\, U(t, x(t))\, dt \leqslant \beta.$$

Letting now $\delta \to 0$:

$$\int_0^T U(t, x(t))\, dt \leqslant \beta.$$

Since U goes to infinity as $x(t) - P_\mu(t)$ or $x(t) - P_\nu(t)$ tends to zero, it is immediate that the set V where $x(t) - P_\mu(t)$ or $x(t) - P_\nu(t)$ eventually vanishes must have measure zero. Moreover, since $x(t) - P_\mu(t)$ and $x(t) - P_\nu(t)$ are continuous functions, the complement of this set ($\mathbf{C}V$) is open.

Let us now take a compact $K = [a, b]$ contained in $\mathbf{C}V$.

Since $x_j \in C^2([0, T])$, $\ddot{x}_j = (\partial/\partial x)\, U_j(t, x_j)$, and $x_{\varepsilon_j} \to x$ weakly in E,

$$\frac{\partial}{\partial x} U_j(t, x_j) \to \frac{\partial}{\partial x} U(t, x) \qquad t \in K.$$

One has

$$\int_K \ddot{x}_j\, y\, dt = \int_K \frac{\partial}{\partial x} U_j(t, x_j)\, y\, dt \qquad \text{for every} \quad y \in E. \tag{4.37}$$

If in particular $y(t) = 0$ for $t \in \partial K$, then (4.37) implies

$$-\int_K \dot{x}_j\, \dot{y}\, dt = \int_K \frac{\partial}{\partial x} U_j(t, x_j)\, y\, dt. \tag{4.38}$$

Set

$$\phi(t) := \frac{\partial}{\partial x} U(t, x(t)) \qquad t \in K.$$

We claim that $\phi(t) = \ddot{x}(t)$ for $t \in K$. Indeed,

$$\int_K \phi(t)\, y(t)\, dt = \int_K \frac{\partial}{\partial x} U(t, x(t))\, y(t)\, dt.$$

Since ϕ is continuous on K, we can consider

$$\Phi(t) = \int_a^t \phi(t)\, dt.$$

Integrating by parts we get

$$\int_K \Phi(t)\, \dot{y}(t)\, dt + \int_K \frac{\partial}{\partial x} U(t, x(t))\, y(t)\, dt = 0 \qquad \text{for all } y \text{ in } E \text{ s.t. } y = 0 \text{ on } \partial K.$$

On the other hand the weak convergence in E of x_j to x, together with (4.38) implies

$$\int_K \dot{x}(t)\, \dot{y}(t)\, dt + \int_K \frac{\partial}{\partial x} U(t, x(t))\, y(t)\, dt = 0 \qquad \text{for all } y \text{ in } E.$$

Hence

$$\int_K (\Phi(t) - \dot{x}(t))\, \dot{y}(t)\, dt = 0 \qquad \text{for all } y \text{ in } E \text{ s.t. } y = 0 \text{ on } \partial K. \tag{4.39}$$

Now:

Φ is continuous on K (actually $\Phi \in C^1(K)$);

$\dot{x}$ is continuous (indeed,

$$\int_K x(t)\,\ddot{z}(t)\,dt = -\int_K \dot{x}(t)\,\dot{z}(t)\,dt = \int_K \frac{\partial}{\partial x} U(t, x(t))\,z(t)\,dt$$

$$\leqslant \left\| \frac{\partial}{\partial x} U(t, x(t)) \right\|_{L^2(K)} \|z(t)\|_{L^2(K)} \qquad \text{for all } z, \text{ s.t. } z=0 \text{ on } \partial K$$

implies $x \in H^{2,2}$ and therefore $x \in C^1(K)$).

Then $\Phi - \dot{x}$ is a continuous constant function on K.

Then $\dot{x} = \Phi - c$, and since $\Phi \in C^1(K)$, $\dot{x}$ is of class $C^1(K)$ and $\ddot{x}(t) = \phi(t)$ for $t \in K$.

Since $\mathbf{C}V$ can be thought of as an union of compact intervals, one has that $x \in C^2(\mathbf{C}V)$. Moreover for every compact set in $\mathbf{C}V$

$$\ddot{x}(t) = \frac{\partial}{\partial x} U(t, x(t)).$$

Hence this equation holds for every $t \in \mathbf{C}V$.

All that suggests the definition of a milder type of solution of (1.3), let us say a generalized solution (this notion has first been introduced in [22] for more general systems).

DEFINITION 4.40. We will call generalized T-periodic solution of (1.3) a continuous loop $x \in C(\mathbb{R}, \mathbb{R}^2)$, s.t. $x(T) = x(0)$ and

(i) $x \in H^{1,2}(\mathbb{R}, \mathbb{R}^2)$ and $I(x) < +\infty$;

(ii) $x(t) - P_\mu(t)$ as well as $x(t) - P_\nu(t)$ vanishes on a set V of measure zero and whose complement is open;

(iii) x is of class C^2 on $\mathbb{R} \backslash V$ and satisfies (1.1) on $\mathbb{R} \backslash V$.

Our goal now is to prove that there exists a multiplicity of T-periodic generalized solutions for the elliptic restricted three body problem, namely to give the

Proof of Theorem 3. Recall that

$$\rho := \min_{0 \leqslant t \leqslant T} |P_\mu(t) - P_\nu(t)|$$

and set

$$R := \max_{0 \leqslant t \leqslant T} |P_\mu(t) - P_\nu(t)|.$$

Denote by H_k $(k \in N^*)$ the homotopy class (in $[0, T] \times \mathbb{R}^2 \setminus \{S \cap ([0, T] \times \mathbb{R}^2)\}$) of continuous curves $(t, x(t))$ (with values in $[0, T] \times \mathbb{R}^2 \setminus \{S \cap ([0, T] \times \mathbb{R}^2)\}$ and $x(T) = x(0)$), which contains in particular the circular loop of components

$$\left(t, 2R \cos \frac{2\pi k}{T} t, 2R \sin \frac{2\pi k}{T} t\right).$$

Also, call $H_k^* := \{x \text{ in } E\colon (t, x(t)) \in H_k\}$. H_k^* is a component of $\Lambda_{(k,k)}$ (Λ_h for $h \in \mathbb{Z}^N$ is defined in (2.9)).

The first step in our proof of Theorem 3 is the following

LEMMA 13. *For every curve y in H_k^**

$$\text{arc length}(y)\,|_{[0,T]} \geqslant 2\rho(k-1) - L, \tag{4.41}$$

where $L = \text{arc length}(P_\mu)\,|_{[0,T]}$.

Proof. Let us write

$$y(t) = y(t) - P_\mu(t) + P_\mu(t).$$

If

$$\eta(t) := y(t) - P_\mu(t),$$

then

$$|\dot{y}(t)| \geqslant |\dot{\eta}(t)| - |\dot{P}_\mu(t)|.$$

As a consequence,

$$\text{arc length}(y) \geqslant \text{arc length}(\eta) - \text{arc length}(P_\mu). \tag{4.42}$$

In a coordinate system centered at $P_\mu(t)$ during an interval of time of length T the loop y winds k times around the origin $(\equiv P_\mu(t))$, while $P_\nu(t)$ winds once around it. Hence there exist instants

$$\tau_1 < \tau_2 < \cdots < \tau_k = \tau_1 + T < \tau_{k+1}$$

s.t. $y(\tau_i)$ lies on the straight line passing through 0 $(\equiv P_\mu(t))$ and $P_\nu(\tau_i)$ and more specifically $P_\nu(\tau_i)$ belongs to the open line segment between 0 and $y(\tau_i)$. Also, there exist instants σ_i, ξ_i, $i = 1, \ldots, k$ s.t.

$$\tau_i < \sigma_i < \xi_i < \tau_{i+1}$$

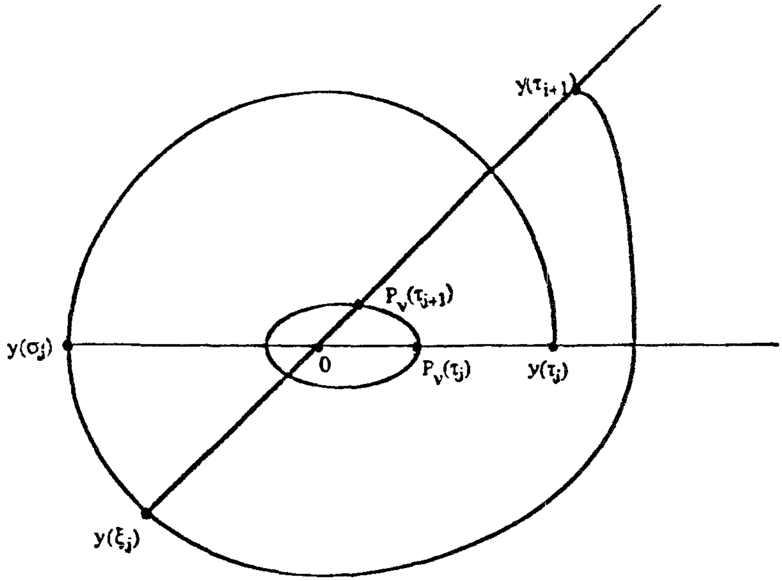

FIG. 1. Geometrical description of the times τ_i, σ_i, ξ_i.

and

(1) $y(\sigma_i)$ lies on the straight line passing through 0 ($\equiv P_\mu(t)$) and $P_\nu(\tau_i)$, and 0 ($\equiv P_\mu(t)$) is between $y(\sigma_i)$ and $P_\nu(\tau_i)$;

(2) $y(\xi_i)$ lies on the straight line passing through 0 ($\equiv P_\mu(t)$) and $P_\nu(\tau_{i+1})$, and 0 ($\equiv P_\mu(t)$) is between $y(\xi_i)$ and $P_\nu(\tau_{i+1})$.

Since η is a periodic function, arc length$(\eta)|_{[0,T]}$ can be evaluated on any interval of length T. We consider the interval given as $\bigcup_{i=1}^{k-1} [\tau_i, \tau_{i+1}]$ and we prove next that the length of the curve $\eta(t)$ in any interval $[\tau_i, \tau_{i+1}]$ is greater or equal to 2ρ. Indeed,

$$\begin{aligned} \text{arc length}(\eta)|_{[\tau_i,\tau_{i+1}]} &\geqslant \int_{\tau_i}^{\sigma_i} |\dot\eta(t)|\,dt + \int_{\xi_i}^{\tau_{i+1}} |\dot\eta(t)|\,dt \\ &\geqslant |\eta(\sigma_i)-\eta(\tau_i)| + |\eta(\tau_{i+1})-\eta(\xi_i)| \\ &\geqslant |y(\sigma_i)-y(\tau_i)| + |y(\tau_{i+1})-y(\xi_i)| \\ &\geqslant |0-P_\nu(\tau_i)| + |0-P_\nu(\tau_{i+1})| \geqslant 2\rho. \end{aligned}$$

From that and (4.42) the thesis follows. ▮

A further remark is the following:

LEMMA 14. *Let x_j $(j \in \mathbb{N})$ be a sequence of loops in H_k^* uniformly converging to a certain loop $x \in E$. Then*

$$\text{arc length}(x)|_{[0,T]} \geqslant 2\rho(k-1) - L, \tag{4.43}$$

where $L = \text{arc length}(P_\mu)|_{[0,T]}$.

Proof. Let

$$\eta_j(t) = x_j(t) - P_\mu(t) \qquad j \in \mathbb{N}$$

and

$$\eta(t) = x(t) - P_\mu(t).$$

We claim that

$$\text{arc length}(\eta)|_{[0,T]} \geqslant 2\rho(k-1). \tag{4.44}$$

Equation (4.43) will then follow from (4.44) as a consequence of the inequality

$$|\dot{x}(t)| \geqslant |\dot{\eta}(t)| - |\dot{P}_\mu(t)|.$$

We know from Lemma 13 that for every fixed $j \in \mathbb{N}$ there is a collection of instants:

$$0 \leqslant \tau_1(j) < \tau_2(j) < \cdots < \tau_k(j) = \tau_1(j) + T < \tau_{k+1}(j)$$

and $\sigma_i(j)$, $\xi_i(j)$, $i = 1, ..., k$ s.t. for every $i = 1, ..., k$

$$0 \leqslant \tau_i(j) < \sigma_i(j) < \xi_i(j) < \tau_{i+1}(j)$$

and

$$\begin{aligned} |\eta_j(\sigma_i(j)) - \eta_j(\tau_i(j))| &\geqslant \rho \\ |\eta_j(\tau_{i+1}(j)) - \eta_j(\xi_i(j))| &\geqslant \rho. \end{aligned} \tag{4.45}$$

Call

$$\bar{\tau}_i = \lim_{j \to \infty} \tau_i(j), \qquad \bar{\sigma}_i = \lim_{j \to \infty} \sigma_i(j), \qquad \bar{\xi}_i = \lim_{j \to \infty} \xi_i(j). \tag{4.46}$$

These limits exist and satisfy

$$0 \leqslant \bar{\tau}_1 < \bar{\sigma}_1 < \bar{\xi}_1 < \bar{\tau}_2 < \cdots < \bar{\tau}_k = \bar{\tau}_1 + T.$$

In order to prove (4.41) we consider the interval given as $\bigcup_{i=1}^{k-1} [\bar{\tau}_i, \bar{\tau}_{i+1}]$. We prove next that for every $i = 1, ..., k$

$$\text{arc length}(\eta)|_{[\bar{\tau}_i, \bar{\tau}_{i+1}]} \geqslant 2\rho. \tag{4.47}$$

One has

$$\text{arc length}(\eta)|_{[\bar\tau_i,\bar\tau_{i+1}]} \geqslant \int_{\bar\tau_i}^{\bar\sigma_i} |\dot\eta(t)|\,dt + \int_{\bar\xi_i}^{\bar\tau_{i+1}} |\dot\eta(t)|\,dt$$
$$\geqslant |\eta(\bar\sigma_i)-\eta(\bar\tau_i)| + |\eta(\bar\tau_{i+1})-\eta(\bar\xi_i)|. \qquad (4.48)$$

Now, for every $j \in \mathbb{N}$ we can write

$$|\eta(\bar\sigma_i)-\eta(\bar\tau_i)| \geqslant |\eta_j(\sigma_i(j))-\eta_j(\tau_i(j))| - |\eta(\bar\sigma_i)-\eta(\sigma_i(j))| - |\eta(\bar\tau_i)-\eta(\tau_i(j))|$$
$$- |\eta(\sigma_i(j))-\eta_j(\sigma_i(j))| - |\eta(\tau_i(j))-\eta_j(\tau_i(j))|.$$

In particular letting $j \to \infty$ we obtain (in view of (4.45), (4.46), and the continuity of the functions η and η_j)

$$|\eta(\bar\sigma_i)-\eta(\bar\tau_i)| \geqslant \rho.$$

In a similar way we deduce

$$|\eta(\bar\tau_{i+1})-\eta(\bar\xi_i)| \geqslant \rho.$$

This together with (4.48) gives the estimate (4.47). Equation (4.44) (and therefore also (4.43)) then follows immediately. ▮

We finally prove Theorem 3; namely we prove that there is a sequence x_k of distinct generalized solutions of the restricted three body problem.

Consider for every positive integer k ($\neq 0$) and every small, positive ε the loop $x_{\varepsilon,k}$ which minimizes the functional I_ε in H_k^*. The existence of such a minimizing loop is guaranteed by Theorem 1. See indeed the remark following the statement of that theorem. Lemma 12 guarantees the existence of a sequence of loops $x_{\varepsilon_j,k}$ weakly converging in E (and therefore uniformly converging) to an element $x_k \in E$. We are not able to deduce that x_k is in Λ and consequently in H_k^*. But, according to Lemma 14, we deduce

$$\text{arc length}(x_k) \geqslant 2\rho(k-1) - L.$$

Then the estimate

$$I(x_k) \geqslant \frac{1}{2T}(\text{arc length}(x_k))^2 \geqslant \frac{1}{2T}[2\rho(k-1)-L]^2 \quad \text{for all } k \geqslant (L/2\rho)+1$$

implies that infinitely many of the curves x_k ($k \in \mathbb{N}^*$, $k \geqslant (L/2\rho)+1$) are distinct. These x_k's are generalized solutions. Indeed $x_k \in E$ and

$$I(x_k) \leqslant \liminf I(x_{\varepsilon_j,k}) < +\infty.$$

Properties (ii) and (iii) in the definition of the generalized solution have already been proved in the remark following Lemma 12.

This finishes the proof of Theorem 3. ∎

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References

1. C. C. Conley, On some new long periodic solutions of the plane restricted three body problem, *Comm. PAM* **16** (1963), 449–467.
2. V. Szebehely, "Theory of Orbits," Academic Press, Orlando, FL, 1967.
3. C. L. Siegel and J. K. Moser, "Lectures on Celestial Mechanics," Springer-Verlag, Berlin, 1971.
4. J. K. Moser, Stable and random motions in dynamical systems, "Annals of Mathematics Studies," Princeton, 1973.
5. H. Poincaré, Les méthodes nouvelles de la Mécanique Céleste, Vol. 1–3, Gauthier–Villars, Paris, 1892–1899; reprinted by Libraire Scientifique et Technique A. Blanchard, Paris, 1987.
6. L. Euler, De motu rectilineo trium corporum se mutuo attrahentium, *Novi Comm. Acad. Sci. Imp. Petrop.* **11** (1767), 144–151.
7. J. L. Lagrange, *Ouvres* **6** (1873), 272–292.
8. G. D. Birkhoff, The restricted problem of three bodies, *Rend. Circ. Mat. Palermo* **39** (1915), 265–334.
9. C. C. Conley, Twist mappings, linking, analyticity and periodic solutions which pass close to an unstable periodic solution, *in* "Topological Dynamics: An International Symposium" (J. Auslander and W. Gottschalk, Eds.), Benjamin, New York, 1968.
10. C. C. Conley, On the ultimate behavior of orbits with respect to an unstable critical point. I.. Oscillating, asymptotics and capture orbits, *J. Differential Equations* **5** (1969), 136–158.
11. C. C. Conley, Low energy transit orbits in the restricted three body problem, *SIAM J. Appl. Math.* **16** (1968), 732–746.
12. R. P. McGehee, "Some Homoclinic Orbits for the Restricted Three Body Problem," Ph.D. Thesis, University of Wisconsin, 1969.
13. E. T. Whittaker, On periodic orbits, *Monthly Notices Roy. Astronom. Soc.* **62** (1901–1902), 186–193.
14. E. T. Whittaker, On periodic orbits in the restricted problem of three bodies, *Monthly Notices Roy. Astronom. Soc.* **62** (1901–1902), 346–352.
15. A. Signorini, Esistenza di un estremale chiusa dentro un contorno di Whittaker, *Rend. Circ. Mat. Palermo* **33** (1912), 187–193.
16. A. Signorini, Sul teorema di Whittaker, *Rend. R. Accad. Lincei (5)* **21** (1912), 36–39.
17. L. Tonelli, Sulle orbite periodiche, *Rend. R. Accad. Lincei (5)* **21** (1912), 251–258.

18. L. TONELLI, Sui massimi e minimi assoluti del Calcolo delle Variazioni, *Rend. Circ. Mat. Palermo* **32** (1911), 297–337.
19. G. D. BIRKHOFF, Dynamical systems with two degrees of freedom, *Trans. Amer. Math. Soc.* **18** (1917), 199–300.
20. W. B. GORDON, Conservative dynamical systems involving strong forces, *Trans. Amer. Math. Soc.* **204** (1975), 113–135.
21. W. B. GORDON, A minimizing property of Keplerian orbits, *Amer. J. Math.* **99** (1977). 961–971.
22. A. BAHRI AND P. H. RABINOWITZ, A minimax method for a class of Hamiltonian Systems with singular potential, *J. Funct. Anal.* **82** (1989), 412–428.
23. V. COTI ZELATI, Periodic solutions for N-body type problems, preprint, 1989.
24. A. BAHRI, Periodic solutions of some problems of 3-body type, lecture notes, 1989.
25. M. DEGIOVANNI, F. GIANNONI, AND A. MARINO, Dynamical systems with newtonian type potentials, *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (81)* (1987), 271–278.
26. M. DEGIOVANNI AND F. GIANNONI, Nonautonomous perturbations of Newtonian potentials, preprint, 1988.
27. A. AMBROSETTI AND V. COTI ZELATI, Critical points with lack of compactness and singular dynamical systems, *Annali Mat. Pura e Appl.* **149** (1987), 237–259.
28. A. AMBROSETTI AND V. COTI ZELATI, Non collision orbits for a class of Keplerian-like potentials, *Ann. Inst. H. Poincaré* **5** (1988), 287–295.
29. C. GRECO, Periodic solutions of a class of singular Hamiltonian Systems, *Nonlinear Anal. TMA* **12** (1988), 259–269.
30. C. GRECO, Periodic solutions of some nonlinear ODE with singular nonlinear part, *Boll. Un. Mat. Ital. B*, in print.
31. A. CAPOZZI, C. GRECO, AND A. SALVATORE, Lagrangian Systems in the presence of singularities, *Proc. Amer. Math. Soc.* **102** (1988), 125–130.
32. P. H. RABINOWITZ, Minimax methods in critical point theory with applications to differential equations, *in* "CBMS Regional Conference Series in Mathematics," Vol. 65, Amer. Math. Soc., Providence, RI, 1986.